Singular Riemannian Foliations with Sections [*]

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Abstract

A singular foliation on a complete riemannian manifold is said to be riemannian if every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets. In this paper we study singular riemannian foliations that have sections, i.e., totally geodesic complete immersed submanifolds that meet each leaf orthogonally and whose dimensions are the codimensions of the regular leaves.

We prove here that the restriction of the foliation to a slice of a leaf is diffeomorphic to an isoparametric foliation on an open set of an euclidian space. This result gives us local information about the singular foliation and in particular about the singular stratification of the foliation. It also allows us to describe the plaques of the foliation as level sets of a transnormal map (a generalisation of an isoparametric map). We also prove that the regular leaves of a singular riemannian foliation with sections are locally equifocal. We use this property to define a singular holonomy. Then we establish some results about this singular holonomy and illustrate them with a couple of examples.

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1 Introduction

In this section we shall introduce the concept of a singular riemannian foliation with sections, review typical examples of this kind of foliation and state our main results (Theorem 2.7 and Theorem 2.10), which relate the new concept with the concepts of isoparametric and equifocal submanifolds.

We start by recalling the definition of a singular riemannian foliation (see the book of P. Molino [6]).

Definition 1.1 A partition $\mathcal{F}$ of a complete riemannian manifold M by connected immersed submanifolds (the *leaves*) is called *singular riemannian foliation on M* if it verifies the following conditions

1. $\mathcal{F}$ is *singular*, i.e., the set $\mathcal{X}_F$ of smooth vector fields on M that are tangent at each point to the corresponding leaf is transitive on each leaf. In other words, for each leaf L and each $p \in L$, one can find vector fields $v_i \in \mathcal{X}_F$ such that $\{v_i(p)\}$ is a basis of $T_p L$.

2. The partition is *transnormal*, i.e., every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

Let $\mathcal{F}$ be a singular riemannian foliation on an complete riemannian manifold M. A point $p \in M$ is called *regular* if the dimension of the leaf L_p that contains p is maximal. A point is called *singular* if it is not regular. Let L be an immersed submanifold of a riemannian manifold M. A section ξ of the normal bundle $\nu(L)$ is said to be a *parallel normal field* along L if $\nabla^\nu \xi \equiv 0$, where ∇^ν is the normal connection. L is said to have globally flat normal bundle, if the holonomy of the normal bundle $\nu(L)$ is trivial, i.e., if any normal vector can be extended to a globally defined parallel normal field.

Definition 1.2 (s.r.f.s.) Let $\mathcal{F}$ be a singular riemannian foliation on a complete riemannian manifold M. $\mathcal{F}$ is said to be a *singular riemannian foliation with section* (s.r.f.s. for short) if for every regular point p, the set $\sigma := \exp_p(\nu L_p)$ is an immersed complete submanifold that meets each leaf orthogonally and if the regular points of σ are dense in it. σ is called a *section*.

Let $p \in M$ and $\mathrm{Tub}(P_p)$ be a tubular neighborhood of a plaque P_p that contains p. Then the connected component of $\exp_p(\nu P_p) \cap \mathrm{Tub}(P_p)$ that contains p is called a *slice* at p. Let Σ_p denote it. Now consider the intersection

of Tub(P_p) with a section of the foliation. Each connected component of this set is called a *local section*. These two concepts play here an important role and are related to each other. In fact, we show in Proposition 2.1 that *the slice at a singular point is the union of the local sections that contain this singular point.*

Typical examples of singular riemannian foliations with sections are the orbits of a polar action, parallel submanifolds of an isoparametric submanifolds in a space form and parallel submanifolds of an equifocal submanifold with flat sections in a compact symmetric space, concepts that we now recall.

An isometric action of a compact Lie group on a riemannian manifold M is called *polar* if there exists a complete immersed submanifold σ of M that meets each G-orbit orthogonally. Such σ is called a *section*. A typical example of a polar action is a compact Lie group with a biinvariant metric that acts on itself by conjugation. In this case the maximal tori are the sections.

A submanifold of a real space form is called *isoparametric* if its normal bundle is flat and if the principal curvatures along any parallel normal vector field are constant. The history of isoparametric hypersurfaces and submanifolds and their generalizations can be found in the survey [9] of G. Thorbergsson.

Now we recall the concept of an equifocal submanifold that was introduced by C.L. Terng and G. Thorbergsson [8] as a generalization of the concept of an isoparametric submanifold.

Definition 1.3 A connected immersed submanifold L of a complete riemannian manifold M is called *equifocal* if

0) the normal bundle $\nu(L)$ is globally flat,

1) for each parallel normal field ξ along L, the derivative of the map $\eta_\xi : L \to M$, defined as $\eta_\xi(x) := \exp_x(\xi)$, has constant rank,

2) L has sections, i.e., for all $p \in L$ there exists a complete, immersed, totally geodesic submanifold σ such that $\nu_p(L) = T_p\sigma$.

A connected immersed submanifold L is called *locally equifocal* if, for each $p \in L$, there exists a neighborhood $U \subset L$ of p in L such that U is an equifocal submanifold.

Finally we are ready to state our main results.

Theorem 2.7 *The regular leaves of a singular riemannian foliation with sections on a complete riemannian manifold M are locally equifocal. In addition, if all the leaves are compact, then the union of regular leaves that are equifocal is an open and dense set in M.*

This result implies that given an equifocal leaf L we can reconstruct the singular foliation taking all parallel submanifolds of L (see Corollary 2.9). In other words, *let L be a regular equifocal leaf and Ξ denote the set of all parallel normal fields along L. Then $\mathcal{F} = \{\eta_\xi(L)\}_{\xi \in \Xi}$.* Theorem 2.7 allows us to define a singular holonomy. We also establish some results about this singular holonomy (see section 3) and illustrate them with a couple of new examples. Theorem 2.7 is also used to prove the following result:

Theorem 2.10 (slice theorem) *Let $\mathcal{F}$ be a singular riemannian foliation with sections on a complete riemannian manifold M and Σ_q the slice at a point $q \in M$. Then $\mathcal{F}$ restricted to Σ_q is diffeomorphic to an isoparametric foliation on an open set of $\mathbf{R}^n$, where n is the dimension of Σ_q.*

Owing to the slice theorem, we can see the plaques of the singular foliation, which are in a tubular neighborhood of a singular plaque P, as the product of isoparametric submanifolds and P. In particular, we can better understand the singular stratification (see Corollary 2.11).

A consequence of the slice theorem is Proposition 2.12 that claims that *the plaques of a s.r.f.s. are always level sets of a transnormal map,* concept that we recall below.

Definition 1.4 Let M^{n+q} be a complete riemannian manifold. A smooth map $H = (h_1 \cdots h_q) : M^{n+q} \to \mathbf{R}^q$ is called *transnormal* if

0) H has a regular value,

1) for every regular value c there exists a neighborhood V of $H^{-1}(c)$ in M and smooth functions b_{ij} on $H(V)$ such that, for every $x \in V$,
   $< \operatorname{grad} h_i(x), \operatorname{grad} h_j(x) >= b_{ij} \circ H(x)$,

2) there is a sufficiently small neighborhood of each regular level set such that $[\operatorname{grad} h_i, \operatorname{grad} h_j]$ is a linear combination of $\operatorname{grad} h_1 \cdots \operatorname{grad} h_q$, with coefficients being functions of H, for all i and j.

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This definition is equivalent to saying that H has a regular value and for each regular value c there exists a neighborhood V of $H^{-1}(c)$ in M such that $H\mid_V \to H(V)$ is an integrable riemannian submersion, where the metric (g_{ij}) of $H(V)$ is the inverse matrix of (b_{ij}).

A transnormal map H is said to be an *isoparametric map* if V can be chosen to be M and $\triangle h_i = a_i \circ H$, where a_i are smooth functions.

Isoparametric submanifolds in space forms and equifocal submanifolds with flat sections in simply connected symmetric spaces of compact type can always be described as regular level sets of transnormal analytic maps, see R.Palais and C.L.Terng [7] and E. Heintzte, X.Liu and C.Olmos [5].

We prove in [1] that *the regular leaves of an analytic transnormal map on an analytic complete manifold are equifocal submanifolds and leaves of a singular riemannian foliation with sections.* Hence, Proposition 2.12 is a local converse of this result.

This paper is organized as follows. In section 2 we shall prove some propositions about singular riemannian foliation with sections (s.r.f.s. for short), Theorem 2.7 and Theorem 2.10. In section 3 we shall introduce the concept of singular holonomy of a s.r.f.s. and establish some results about it. In section 4 we illustrate some properties of singular holonomies constructing singular foliations by suspensions of homomorphisms.

2   Proof of the main results

Proposition 2.1 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and let $q \in M$. Then*

a) *$\Sigma_q = \cup_{\sigma \in \Lambda(q)}\, \sigma$, where $\Lambda(q)$ is the set of all local sections that contain q.*

b) *$\Sigma_x \subset \Sigma_q$ for all $x \in \Sigma_q$.*

c) *$T_x\Sigma_q = T_x\Sigma_x \oplus T_x(U \cap \Sigma_q)$, where $x \in \Sigma_q$ and $U \subset L_x$ is an open set of x in L_x.*

Proof: a) At first we check that $\Sigma_q \supset \cup_{\sigma \in \Lambda(q)} \sigma$. Let σ be a local section that contains q, let p be a regular point of σ and γ the shortest segment of geodesic that joins q to p. Then γ is orthogonal to L_p for $\gamma \subset \sigma$ and σ is orthogonal to L_p. Since $\mathcal{F}$ is a riemannian foliation, γ is also orthogonal to L_q and hence $p \in \Sigma_q$. Since the regular points are dense in σ, $\Sigma_q \supset \sigma$.

Now we check that $\Sigma_q \subset \cup_{\sigma \in \Lambda(q)} \sigma$. Let $p \in \Sigma_q$ be a regular point and γ the segment of geodesic orthogonal to L_q that joins q to p. Since $\mathcal{F}$ is a riemannian foliation, γ is orthogonal to L_p. Therefore γ belongs to the local section σ that contains p. In particular $q \in \sigma$. In other words, each regular point $p \in \Sigma_q$ belongs to a local section σ that contains q.

Finally let $z \in \Sigma_q$ be a singular point, σ a local section that contains z, and p a regular point of σ. Since the slice is defined on a tubular neighborhood of a plaque P_q, there exists only one point $\tilde{q} \in P_q$ such that $p \in \Sigma_{\tilde{q}}$. As we have shown above, $\tilde{q} \in \sigma$. Now it follows from the first part of the proof that $z \in \sigma \subset \Sigma_{\tilde{q}}$ Since $z \in \Sigma_q$, $\tilde{q} = q$.

b) Let $x \in \Sigma_q$ and $\sigma \subset \Sigma_x$ a local section. It follows from the proof of item a) that $\sigma \subset \Sigma_q$ and $q \in \sigma$. Since Σ_x is a union of local sections that contain x, $\Sigma_x \subset \Sigma_q$.

c) Since the foliation $\mathcal{F}$ is singular, we have:

$$\begin{aligned}
\dim T_x\Sigma_x + \dim T_x(U \cap \Sigma_q) + \dim L_q &= \dim M \\
&= \dim T_x\Sigma_q + \dim L_q.
\end{aligned}$$

The item b) and the above equation imply the item c) □

Remark 2.2 In [6] (pag 209) Molino showed that given a singular riemannian foliation $\mathcal{F}$ it is possible to change the metric in such way that the restriction of $\mathcal{F}$ to a slice is a singular riemannian foliation with respect to this new metric. This change respect the distance between the leaves. As we see below this change is not necessary if the singular riemannian foliation has sections.

Corollary 2.3 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and Σ a slice. Then $\mathcal{F} \cap \Sigma$ is a s.r.f.s. on Σ with the induced metric of M.*

Proof: Let γ be a segment of geodesic that is orthogonal to L_x where $x \in \Sigma$. Since $\gamma \subset \Sigma_x$, it follows from the item b) of Proposition 2.1 that $\gamma \subset \Sigma$. Since

$\mathcal{F}$ is riemannian, γ is orthogonal to the leaves of $\mathcal{F} \cap \Sigma$. Therefore $\mathcal{F} \cap \Sigma$ is a singular riemannian foliation.

Now let σ be a local section that contains x. Then it follows from item a) of Proposition 2.1 that $\sigma \subset \Sigma$ and hence σ is a local section of $\mathcal{F} \cap \Sigma$. Therefore $\mathcal{F} \cap \Sigma$ is a s.r.f.s. $\square$

Proposition 2.4 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and γ a geodesic orthogonal to the leaves of $\mathcal{F}$. Then the singular points of γ are either all the points of γ or isolated points of γ.*

Proof: Since the set of regular points on γ is open, we can suppose that $q = \gamma(0)$ is a singular point and that $\gamma(t)$ is a regular point for $-\delta < t < 0$. We shall show that there exists $\epsilon > 0$ such that $\gamma(t)$ is also a regular point for $0 < t < \epsilon$.

At first we note that we can choose $t_0 < 0$ such that q is a focal point of $L_{\gamma(t_0)}$. To see this let $\mathrm{Tub}(P_q)$ be a tubular neighborhood of a plaque P_q and $t_0 < 0$ such that $\gamma(t_0) \in \mathrm{Tub}(P_q)$. Since $L_{\gamma(t_0)}$ is a regular leaf and q is a singular point, it follows from item c) of Proposition 2.1 that $L_{\gamma(t_0)} \cap \Sigma_q$ is not empty. Then we can join this submanifold to q with geodesics that belong to Σ_q. Since $\mathcal{F}$ is a riemannian foliation, these geodesics are also orthogonal to $L_{\gamma(t_0)} \cap \Sigma_q$. This implies that q is a focal point.

Since focal points are isolated along γ, we can choose $\epsilon > 0$, such that $\gamma(t)$ is not a focal point of $P_{\gamma(t_0)}$ along γ for $0 < t < \epsilon$.

Suppose there exists $0 < t_1 < \epsilon$ such that $x = \gamma(t_1)$ is a singular point. Let σ a local section that contains $\gamma(t_0)$. Let U an open set of $\nu_x L$ such that $\widetilde{\Sigma}_x := \exp_x(U)$ contains $\gamma(t_0)$ and is contained in a convex neighborhood of x. We note that $\widetilde{\Sigma}_x$ is not contained in a tubular neighborhood of P_x and hence is not a slice.

We have:
$$\sigma \subset \widetilde{\Sigma}_x.$$

Since x is a singular point, we have:
$$\dim \sigma < \dim \widetilde{\Sigma}_x.$$

The equations above implies that $\dim P_{\gamma(t_0)} \cap \widetilde{\Sigma}_x > 0$. Hence we can find geodesics in $\widetilde{\Sigma}_x$ that join x to the submanifold $P_{\gamma(t_0)} \cap \widetilde{\Sigma}_x$. Since the foliation is a riemannian foliation, these geodesics are also orthogonal to $P_{\gamma(t_0)} \cap \widetilde{\Sigma}_x$

and hence x is a focal point of this submanifold. This contradicts our choice of ϵ and completes the proof. $\square$

In what follows we shall need a result of Heintze, Liu and Olmos.

Proposition 2.5 (Heintze, Liu and Olmos [5]) *Let M be a complete riemannian manifold, L be an immersed submanifold of M with globally flat normal bundle and ξ be a normal parallel field along L. Suppose that $\sigma_x := \exp_x(\nu_x L)$ is a totally geodesic complete submanifold for all $x \in L$, that means, L has sections. Then*

1. *$d\eta_\xi(v)$ is orthogonal to σ_x at $\eta_\xi(x)$ for all $v \in T_x L$.*

2. *Suppose that p is not a critical point of the map η_ξ. Then there exists a neighborhood U of p in L such that $\eta_\xi(U)$ is an embedded submanifold, which meets σ_x orthogonally and has globally flat normal bundle. In addition, a parallel normal field along U transported to $\eta_\xi(U)$ by parallel translation along the geodesics $\exp(t\xi)$ is a parallel normal field along $\eta_\xi(U)$.*

Let $\exp^\perp$ denote the restriction of $\exp$ to $\nu(L)$. We recall that for each $w \in T_{\xi_0}\nu(L)$ there exists only one $w_{\mathsf{t}} \in T_{\xi_0}\nu(L)$ (the tangential vector) and one $w_{\mathsf{n}} \in T_{\xi_0}\nu(L)$ (the normal vector) such that

1. $w = w_{\mathsf{t}} + w_{\mathsf{n}}$,

2. $d\Pi(w_{\mathsf{n}}) = 0$, where $\Pi : \nu(L) \to L$ is the natural projection,

3. $w_{\mathsf{t}} = \xi'(0)$, where $\xi(t)$ is the normal parallel field with $\xi(0) = \xi_0$.

We also recall that $z = \exp^\perp(\xi_0)$ is a focal point with multiplicity k along $\exp^\perp(t\,\xi_0)$ if and only if $\dim \ker d\exp^\perp_{\xi_0} = k$. We call z a focal point of L of *tangencial type* if $\ker d\exp^\perp_{\xi_0}$ only consists of tangential vectors.

Corollary 2.6 *Let L be a submanifold defined as above, $p \in L$ and $\xi_0 \in \nu_p L$. Suppose that the point $z = \exp_p(\xi_0)$ is a focal point of L along $\exp_p(t\,\xi_0)$ that belongs to a normal neighborhood of p. Then z is a focal point of tangencial type.*

Proof: If z is a focal point, then there exists $w \in T_{\xi_0}\nu(L)$ such that $\|d\exp^\perp_{\xi_0}(w)\| = 0$. It follows from the above proposition that

$$< d\exp^\perp_{\xi_0} w_{\mathsf{n}}, d\exp^\perp_{\xi_0} w_{\mathsf{t}} >_{\exp^\perp(\xi_0)} = 0$$

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and hence $\|d\exp_{\xi_0}^{\perp}(w_{\mathsf{n}})\| = 0$. Since z belongs to a normal neighborhood, w_{n} must be zero. We conclude that $w = w_{\mathsf{t}}$. $\square$

Now we can show one of our main results.

Theorem 2.7 *Let $\mathcal{F}$ be a singular riemannian foliation with sections on a complete riemannian manifold M. Then the regular leaves are locally equifocal. In addition, if all the leaves are compact, then the union of regular leaves that are equifocal is an open and dense set in M.*

To prove it, we need the following lemma.

Lemma 2.8 *Let $\mathrm{Tub}(P_q)$ be a tubular neighborhood of a plaque P_q, $x_0 \in \mathrm{Tub}(P_q)$, and $\xi \in \nu P_{x_0}$ such that $\exp_{x_0}(\xi) = q$. We also suppose that q is the only singular point on the segment of geodesic $\exp_{x_0}(t\,\xi) \cap \mathrm{Tub}(P_q)$. Then we can find a neighborhood U of x_0 in P_{x_0} with the following properties:*

1) *νU is globally flat and we can define the parallel normal field ξ on U.*

2) *There exists a number $\epsilon > 0$ such that, for each $x \in U$, $\gamma_x \subset \mathrm{Tub}(P_q)$, where $\gamma_x(t) := \exp_x(t\,\xi)$ and $t \in [-\epsilon, 1+\epsilon]$.*

3) *The regular points of the foliation $\mathcal{F}|_{\mathrm{Tub}(P_q)}$ are not critical values of the maps $\eta_{t\,\xi}|_U$.*

4) *$\eta_{t\,\xi}(U) \subset L_{\gamma_{x_0}(t)}$.*

5) *$\eta_{t\,\xi} : U \to \eta_{t\,\xi}(U)$ is a local diffeomorphism for $t \neq 1$.*

6) *$\dim\mathrm{rank}\,D\eta_{\xi}$ is constant on U.*

Proof: The item 1) follows from the fact that $\mathcal{F}$ has sections and one can show 2) with standard arguments.

3) Let $p = \eta_{r\,\xi}(x_1)$ be a regular point of the foliation and suppose that x_1 is a critical point of the map $\eta_{r\,\xi}|_U$. Then there exists a Jacobi field $J(t)$ along the geodesic γ_{x_1} such that $J(r) = 0$. In particular there exists a smooth curve $\beta(t) \subset P_{x_0}$ such that $J(t) = \frac{\partial}{\partial s}\exp_{\beta(s)}(t\,\xi)$ and $\beta(0) = x_1$.

Since focal points are isolated along $\gamma_{x_1}(t)$, there exists a regular point of the foliation $\tilde{p} = \gamma_{x_1}(\tilde{r})$ that is not a focal point of P_{x_1} along γ_{x_1}. It follows from Proposition 2.5 that there exists a neighborhood V of x_1 in P_{x_0} such that the embedded submanifold $\eta_{\tilde{r}\,\xi}(V)$ is orthogonal to the sections that it

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meets. Hence $\eta_{\tilde{r}\,\xi}(V)$ is tangent to the plaques near to $P_{\tilde{p}}$. Since $\eta_{\tilde{r}\,\xi}(V)$ has the dimension of the regular leaves, $\eta_{\tilde{r}\,\xi}(V)$ is an open subset of $P_{\tilde{p}}$.

Since we can choose $\tilde{p}$ so close to p as necessarily, we can suppose that p and $\tilde{p}$ belong to a neighborhood W that contains only regular points of the foliation and such that $\mathcal{F}|_W$ are pre image of an integrable riemannian submersion $\pi : W \to B$. It follows from Proposition 2.5 that $\gamma'_{\beta(s)}(\tilde{r})$ is a parallel field along the curve $\eta_{\tilde{r}\,\xi} \circ \beta(s) \subset \eta_{\tilde{r}\,\xi}(V) \subset P_{\tilde{p}}$. Therefore $\gamma_{\beta(s)}(t) \cap W$ are horizontal lift of a geodesic in B (the basis of the riemannian submersion π). This implies that $J(r) \neq 0$ This contradicts the assumption that p is a focal point and completes the proof of item 3).

4) At first we check the item 4) for each $t \neq 1$. Fix a $t_0 \neq 1$ and define $K := \{k \in U \text{ such that } \eta_{t_0\,\xi}(k) \in P_{\gamma_{x_0}(t_0)}\}$. Since $\gamma_{x_0}(t_0)$ is a regular point of the foliation, it follows from the item 3) that all the points of $P_{\gamma_{x_0}(t_0)}$ are regular values of the map $\eta_{t_0\,\xi}$. Hence for each $k \in K$ there exists a neighborhood V of k in U such that $\eta_{t_0\,\xi}(V)$ is an embedded submanifold. As we have note in the proof of item 3), $\eta_{t_0\,\xi}(V)$ is an open set of $P_{\gamma_{x_0}(t_0)}$, because this embedded submanifold is orthogonal to the sections and has the same dimension of the plaques. We conclude then that K is an open set. One can prove that K is closed using standard arguments and the fact that the plaques are equidistant. Since U is connected, $K = U$.

Now we check the item 4) for $t = 1$. We define $f(x,t) := d(\eta_{t\,\xi}(x), P_q) - d(\eta_{t\,\xi}(x_0), P_q)$. As we have seen above $\eta_{t\,\xi}(x)$ and $\eta_{t\,\xi}(x_0)$ belong to the same plaque, for $t \neq 1$. This means that $f(x,t) = 0$ for all $t \neq 1$ and hence $f(x,1) = 0$, i.e., $\eta_\xi(x) \subset P_q$.

5) The item 5) follows from the item 3) and 4).

6) Fix a point $x_1 \in U$. It follows from Corollary 2.6 that the focal points of U along $\gamma_x(t)$ are of tangential type. This means that $\gamma_x(t_0)$ is a focal point of U along γ_x with multiplicity k if and only if x is a critical point of $\eta_{t_0\,\xi}$ and $\dim \ker d\eta_{t_0\,\xi}(x) = k$. In addition, it follows from the item 5) that the map $\eta_{t\,\xi}$ might not be a diffeomorphism only for $t = 1$. Therefore we have

$$m(\gamma_x) = \dim \ker d\eta_\xi(x), \tag{1}$$

where $m(\gamma_x)$ denote the number of focal points on $\gamma_x(t)$, each counted with its multiplicities.

On the other hand, we have

$$m(\gamma_x) \geq m(\gamma_{x_1}) \tag{2}$$

for x in a neighborhood of x_1 in U. Indeed one can argue like Q.M. Wang[10] to see that equation 2 follows from the Morse index theorem.

Equations (1) and (2) together with the elementary expression $\dim \ker d\eta_\xi(x) \leq \dim \ker d\eta_\xi(x_1)$ imply that $\dim \ker d\eta_\xi$ is constant on a neighborhood of x_1 in U. Since this hold for each $x_1 \in U$, we conclude that that $\dim \ker d\eta_\xi$ is constant on U. $\square$

Proof of Theorem 2.7

Let L be a leaf of $\mathcal{F}$, U be a open set of L that has normal bundle globally flat and ξ a parallel normal field along U. At first, we shall prove that $\dim \operatorname{rank} d\eta_\xi|_U$ is constant, i.e., L is locally equifocal.

Let $p \in U$. Since singular points are isolated along $\gamma_p(t) = \exp_p(t\,\xi)|_{[-\epsilon,1+\epsilon]}$ (see Proposition 2.4), we can cover this arc of geodesic with a finite number of tubular neighborhood $\operatorname{Tub}(P_{\gamma_p(t_i)})$, where $t_0 = 0$ and $t_n = 1$.

Let $P_{\gamma_p(r_i)}$ be regular plaques that belong to $\operatorname{Tub}(P_{\gamma_p(t_{i-1})}) \cap \operatorname{Tub}(P_{\gamma_p(t_i)})$, where $t_{i-1} < r_i < t_i$. Applying the Lemma 2.8 and Proposition 2.5, we can find an open set $U_0 \subset P_p$, of the plaque P_p, open sets $U_i \subset P_{\gamma_p(r_i)}$ of the plaques $P_{\gamma_p(r_i)}$ and parallel normal fields ξ_i along U_i, with the following properties:

1) For each U_i, the parallel normal field ξ_i is tangent to the geodesics $\gamma_x(t)$, where $x \in U_0$;

2) $\eta_{\xi_i} : U_i \to U_{i+1}$ is a local diffeomorphism for $i < n$;

3) $\eta_\xi|_{U_0} = \eta_{\xi_n} \circ \eta_{\xi_{n-1}} \circ \cdots \circ \eta_{\xi_0}|_{U_0}$

Since $\dim \operatorname{rank} d\eta_{\xi_i}$ is constant on U_i, $\dim d\eta_\xi$ is constant on U_0. Since this hold for each $p \in U$, $\dim d\eta_\xi$ is constant on U, i.e., L is locally equifocal.

At last we check what happens when the leaves of the foliation are compact. According to Molino (Proposition 3.7, page 95 [6]), the union of the regular leaves with trivial holonomie of a singular riemannian foliation is an open and dense set in the set of the regular points. In addition, the set of regular points is an open and dense set in M (see page 197 [6]). Since the leaves of a s.r.f.s. that have trivial holonomie are exactly the leaves that have normal bundle globally flat, the union of regular leaves that are equifocal is an open and dense set in M. $\square$

Corollary 2.9 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and L be a regular leaf of $\mathcal{F}$.*

a) Let $\beta(s) \subset L$ be a smooth curve and ξ a parallel normal field along $\beta(s)$. Then the curve $\exp_{\beta(s)}(\xi)$ belongs to a leaf of the foliation.

b) Let L be a regular equifocal leaf and Ξ denote the set of all parallel normal fields along L. Then $\mathcal{F} = \{\eta_\xi(L)\}_{\xi \in \Xi}$.

Proof: a) The item a) can be easily proved using the item 4) of Lemma 2.8 and gluing tubular neighborhoods as we have already done in the proof of Theorem 2.7.

b) **Statement** Let $x_0 \in L$ and $q = \eta_\xi(x_0)$. Then there exists a neighborhood $U \subset L$ of x_0 in L such that $\eta_\xi(U) \subset P_q$ is an open set in L_q.

To check this statement is enough to suppose that $x_0 \in \text{Tub}(P_q)$, for the general case can be proved gluing tubular neighborhoods as we have done in proof of Theorem 2.7. Now the statement follows if we note that $\eta_\xi : U \to P_q$ is a submersion whose fibers are the intersections of U with the slices of P_q. To see that each fiber is contained in $U \cap \Sigma$ one can use the fact that the rank of $d\eta_\xi$ is constant and the fact that the foliation is riemannian. To see that each fiber contains $U \cap \Sigma$ one can use item 4) of Lemma 2.8 together with item a) of Proposition 2.1.

It follows from the above statement that $\eta_\xi(L)$ is an open set in L_q. We shall see that $\eta_\xi(L)$ is also a closed set in L_q. Let $z \in L_q$ and $\{z_i\}$ a sequence in $\eta_\xi(L)$ such that $z_i \to z$.

At first suppose that L_q is a regular leaf. If follows from Proposition 2.5 that there exists a parallel normal field $\hat{\xi}$ along $\eta_\xi(L)$ such that $\hat{\xi}_{\eta_\xi(x)}$ is tangent to the geodesic $\exp_x(t\,\xi)$. Since the normal bundle of P_z is globally flat, we can extend $\hat{\xi}$ along P_z. The item 4) of Lemma 2.8 implies that $\eta_{-\hat{\xi}} : P_z \to L$. By construction $\eta_\xi \circ \eta_{-\hat{\xi}}(z_i) = z_i$. Therefore $\eta_\xi \circ \eta_{-\hat{\xi}}(z) = z$. This means that $z \in \eta_\xi(L)$.

At last suppose that L_q is a singular leaf. There exists $x_i \in L$ such that $z_i = \eta_\xi(x_i) \in P_z$. We can find a $s < 1$ such that $y_i = \eta_{s\,\xi}(x_i)$ is a regular point. Since y_i is a regular point, the plaque P_{y_i} is an open set of $\eta_{s\,\xi}(L)$ as we have proved above. There exists a parallel normal field $\hat{\xi}$ along P_{y_i} such that $\eta_{\hat{\xi}} \circ \eta_{s\,\xi} = \eta_\xi$.

It follows from item 4) Lemma 2.8 that $\eta_{\hat{\xi}}(P_{y_i}) \subset P_z$. On the other hand, since the foliation is singular, the plaque P_{y_i} intercept the slice Σ_z. These two facts imply that $z \in \eta_{\hat{\xi}}(P_{y_i})$. Therefore $z \in \eta_\xi(L)$. $\square$

Let $\mathcal{F}$ be a foliation on a manifold M^n, $\widetilde{\mathcal{F}}$ a foliation on a manifold $\widetilde{M}^n$ and $\varphi : M \to \widetilde{M}$ a diffeomorphism. We say that φ is a *diffeomorphism between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$* if each leaf L of $\mathcal{F}$ is diffeomorphic to a leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}$.

Theorem 2.10 (slice theorem) *Let $\mathcal{F}$ be a singular riemannian foliation with sections on a complete riemannian manifold M and Σ_q a slice at a point $q \in M$. Then $\mathcal{F}$ restrict to Σ_q is diffeomorphic to an isoparametric foliation on an open set of $\mathbf{R}^n$, where n is the dimension of Σ_q.*

Proof: According to H. Boualem (see Proposition 1.2.3 and Lemma 1.2.4 of [3]), we have

1) the map $\exp^{-1}$ is a diffeomorphim between the foliation $\mathcal{F}|_{\Sigma_q}$ and a singular riemannian foliation with sections $\widetilde{F}$ on an open set of the inner product space $(T_q\Sigma_q, <, >_q)$, where $<, >_q$ denote the metric of $T_q M$,

2) the sections of the singular foliation $\widetilde{F}$ are the vector subspaces $\exp^{-1}(\sigma)$, where σ are the local section of $\mathcal{F}$.

Let $<, >_0$ denote the canonical euclidian product. Then there exists a positive definite symmetric matrix A such that $< X, Y >_q = < A X, Y >_0$ The isometry $\sqrt{A} : (T_q\Sigma_q, <, >_q) \to (\mathbf{R}^n, <, >_0)$ is a diffeomorphismus between the foliation $\widetilde{F}$ and a singular riemannian foliation with section $\widehat{F}$ on an open set of the inner product space $(\mathbf{R}^n, <, >_0)$. Since $0 \in \mathbf{R}^n$ is a singular leaf of the foliation $\widehat{\mathcal{F}}$, the leaves of this foliation belong to spheres in the euclidian space.

Statement 1: The restriction the foliation $\widehat{\mathcal{F}}$ to a sphere $\mathbf{S}^{n-1}(r)$ is a singular riemannian foliation with sections on $\mathbf{S}^{n-1}(r)$.

The first step to check this statement is to note that $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$ is a singular foliation, for $\widehat{\mathcal{F}}$ is a singular foliation. Next we have to note that if $\widehat{\sigma}$ is a section of $\widehat{F}$ then $\sigma_s := \widehat{\sigma} \cap \mathbf{S}^{n-1}(r)$ is a section of the foliation $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$. To conclude, we have to note that $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$ is a transnormal system. Let γ be a geodesic of $\mathbf{S}^{n-1}(r)$ that is orthogonal to a leaf $L_{\gamma(0)}$ of $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$. Since a slice of $\widehat{F}$ is a union of sections, γ is tangent to a section $\widehat{\sigma}$ at the point $\gamma(0)$ and hence is tangent to a section σ_s of $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$ at the point $\gamma(0)$. This implies that $\gamma \subset \sigma_s$, which means that γ is orthogonal to each leaf that it meets, i.e., the partition is transnormal.

Now Theorem 2.7 guarantees that the leaves of a singular riemannian foliation with sections are locally equifocal. Therefore the leaves of $\widehat{\mathcal{F}}|_{\mathbf{S}^{n-1}(r)}$ are locally equifocal.

The next statement follows from standard calculations on space forms.

Statement 2: The locally equifocal submanifolds in $\mathbf{S}^{n-1}(r)$ are isoparametric submanifold in $\mathbf{S}^{n-1}(r)$.

Since isoparametric submanifold in spheres are isoparametric submanifolds in euclidian spaces (see Palais and Terng, Proposition 6.3.17 [7]), we can conclude that the regular leaves of $\widehat{\mathcal{F}}$ are isoparametric submanifold in an open set of the euclidian space $\mathbf{R}^n$.

At last, we note that Corollary 2.9 implies that the singular leaves of $\widehat{\mathcal{F}}$ are the focal leaves. Therefore $\widehat{\mathcal{F}}$ is an isoparametric foliation on an open set of the euclidian space and this completes the proof of the theorem. $\square$

Corollary 2.11 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and σ be a local section contained in a slice Σ_q of dimension n. According to the slice theorem there exist an open set $U \subset \mathbf{R}^n$ and a diffeomorphism $\Psi : \Sigma_q \to U$ that sends the foliation $\mathcal{F} \cap \Sigma_q$ to an isoparametric foliation $\widehat{\mathcal{F}}$ on U. Then the set of singular points of $\mathcal{F}$ contained in σ is a finite union of totally geodesic hypersurfaces that are sent by Ψ onto the focal hyperplanes of $\widehat{\mathcal{F}}$ contained in a section of this isoparametric foliation.*

We shall call *singular stratification of the local section σ* this set of singular points of $\mathcal{F}$ contained in σ.

Proof: It follows from Molino [6](page 194, Proposition 6.3) that the intersection of the singular leaves with a section is a union of totally geodesic submanifolds. Now the slice theorem implies that these totally geodesic submanifolds are in fact hypersurfaces that are diffeomorphic to focal hyperplanes. $\square$

Proposition 2.12 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and $q \in M$. Then there exist a tubular neighborhood $\mathrm{Tub}(P_q)$, an open set $W \subset \mathbf{R}^k$ and a transnormal map $H : \mathrm{Tub}(P_q) \to W$ such that the preimages of H are leaves of the singular foliation $\mathcal{F}|_{\mathrm{Tub}(P_q)}$. The leaf $H^{-1}(c)$ is regular if and only if c is a regular value.*

Proof: We start recalling a result that can be found in the book of Palais and Terng.

Lemma 2.13 (Theorem 6.4.4. page 129 [7]) *Let N be a rank k isoparametric submanifold in $\mathbf{R}^n$, W the associated Coxeter group, q a point on N, $\nu_q = q + \nu(N)$ the affine normal plane at q and $u_1, \cdots, u_k$ be a set of generators of the W-invariant polynomials on ν_q. Then $u = (u_1, \cdots, u_k)$ extends uniquely to an isoparametric polynomial map $g : \mathbf{R}^n \to \mathbf{R}^k$ having N as a regular level set. Moreover,*

1) each regular set is connected,

2) the focal set of N is the set of critical points of g,

3) $\nu_q \cap N = W \cdot q$,

4) $g(\mathbf{R}^n) = u(\nu_q)$,

5) for $x \in \nu_q$, $g(x)$ is a regular value if and only if x is W-regular,

6) $\nu(N)$ is globally flat.

The above result implies that the leaves of the isoparametric foliation, which has N as a leaf, can be described as pre image of a map g. Note that this is even true if N is not a full isoparametric submanifold of $\mathbf{R}^n$.

Now we define $\widetilde{H} : \Sigma_q \to \mathbf{R}^k$ as $\widetilde{H} := g \circ \Psi$, where $\Psi : \sigma_q \to \mathbf{R}^n$ is the diffeomorphism given by the slice theorem that sends $\mathcal{F}|_{\Sigma_q}$ to an isoparametric foliation on an open set W of $\mathbf{R}^n$.

Since $\mathcal{F}$ is a singular foliation, there exists a projection $\Pi : \mathrm{Tub}(P_q) \to \Sigma_q$ such that $\Pi(P) = P \cap \Sigma_q$ for each plaque P.

Finally we define $H := \widetilde{H} \circ \Pi$. Then the preimages of H are leaves of the foliation $\mathcal{F}|_{\mathrm{Tub}(P_q)}$.

The statement below, which can be found in Molino [6][page 77], implies that H is a transnormal map.

Statement Let U a simple neighborhood of a riemannian foliation (with section) and $H : U \to \widetilde{U} \subset \mathbf{R}^k$ such that $H^{-1}(c)$ are leaves of $\mathcal{F}|_U$. Then we can choose a metric for $\widetilde{U}$ such that $H : U \to \widetilde{U}$ is a (integrable) riemannian submersion. $\square$

3   Singular Holonomy

The slice theorem give us a description of the plaques of a singular riemannian foliation with sections. However, it doesn't assure us if two different plaques

belong to the same leaf. To get such kind of information, we must extend the concept of holonomy to describe not only what happens near a regular leaf but also what happens in a neighborhood of a singular leaf.

In this section, we shall introduce the concept of singular holonomy and establish some of its properties.

Proposition 3.1 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M, L_p a regular leaf, σ a local section and $\beta(s) \subset L_p$ a smooth curve, where $p = \beta(0)$ and $\beta(1)$ belong to σ. Let $[\beta]$ denote the homotopy class of β. Then there exists an isometry $\varphi_{[\beta]} : U \to W$, where the source U and target W contain σ, which has the following properties:*

1) $\varphi_{[\beta]}(x) \in L_x$ for each $x \in \sigma$,

2) $d\varphi_{[\beta]}\xi(0) = \xi(1)$, where $\xi(s)$ is a parallel normal field along $\beta(s)$.

Proof: Since σ is a local section, for each $x \in \sigma$, there exists only one $\xi \in T_p\sigma$ such that $\exp_p(\xi) = x$. Let $\xi(t)$ be the parallel transport of ξ along β and define $\varphi_\beta(x) := \exp_{\beta(1)}(\xi(1))$. It's easy to see that φ_β is a bijection. It follows from Corollary 2.9 that $\exp_\beta(\xi) \subset F_x$ and this proves a part of item 1. Since φ_β is an extension of the holonomy map, $d\varphi_\beta\xi(0) = \xi(1)$, and this proves a part of item 2. The fact that φ_β is an extension of the holonomy map implies that the restriction of φ_β to a small neighborhood of σ depend only on the homotopy class of β. Since isometries are determined by the image of a point and the derivative at this point, is enough to prove that φ_β is an isometry to see that φ_β depends only of the homotopy class of β. To see that φ_β is an isometry it's enough to check the following statement.

  Statement Given a point $x_0 \in \sigma$ there exists an open set $V \subset \sigma$ of x_0 in σ such that $d(x_1, x_0) = d(\varphi_\beta(x_1), \varphi_\beta(x_0))$, for each $x_1 \in V$.

  To check the statement let $\xi_0(s)$ and $\xi_1(s)$ be normal parallel fields along $\beta(s)$ such that $x_j = \exp_p(\xi_j(0))$ for $j = 0, 1$. Define $\alpha_j(s) = \exp_{\beta(s)}(\xi_j(s))$ for $j = 0, 1$. Since $\varphi_\beta(x_j) = \alpha_j(1)$ the statement follows from the following equation

$$d(\alpha_0(s), \alpha_1(s)) = d(\alpha_0(0), \alpha_1(0))$$

and this equation follows from the following facts:

1. $\alpha_j(s) \in L_{x_j}$

2. singular riemannian foliations are locally equidistant,

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3. $\alpha_0(s)$ and $\alpha_1(s)$ are always in the same local section. $\square$

Definition 3.2 The pseudosubgroup of isometries generated by the isometries constructed above is called *pseudogroup of singular holonomy of the local section σ*. Let $\mathrm{Holsing}(\sigma)$ denote this pseudogroup.

Proposition 3.3 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M and σ a local section. Then the reflections in the hypersurfaces of the singular stratification of the local section σ let $\mathcal{F} \cap \sigma$ invariant. Moreover these reflections are elements of $\mathrm{Holsing}(\sigma)$.*

Proof: The proposition is already true if the singular foliation is an isoparametric foliation on an euclidean space. In what follows we shall use this fact and the slice theorem to construct the desired reflections.

Let S be a complete totally geodesic hypersurface of the singular stratification of the local section σ and Σ be a slice of a point of S and hence that contains σ. It follows from the slice theorem that there exists a diffeomorphism $\Psi : \Sigma \to V \subset \mathbf{R}^n$ that sends $\mathcal{F} \cap \Sigma$ to an isoparametric foliation $\tilde{F}$ on an open set V of $\mathbf{R}^n$. Let $p \in \sigma$ be a regular point, $\tilde{L} := \Psi(L_p \cap \Sigma)$ and $\tilde{\sigma} := \Psi(\sigma)$. We note that $\tilde{\sigma}$ is a local section of the isoparametric foliation $\tilde{F}$.

It follows from Corollary 2.11 and from the theory of isoparametric submanifolds [7] that $\tilde{S} := \Psi(S)$ is a focal hyperplane associated to a curvature distribution E. Let $\beta \subset \Sigma \cap \mathcal{F}$ with $\beta(0) = p$ and $\beta(1) \in \sigma$ such that $\tilde{\beta} := \Psi \circ \beta$ is tangent to the distribution E. Finally let $z \in S$, $\xi \in T_p\sigma$ such that $\exp_p(\xi) = z$ and $\xi(s)$ the parallel transport of ξ along β.

Statement $\exp_{\beta(s)}(\xi) = z$.

To check this statement, we recall that $\tilde{S} \subset \tilde{\sigma}_{\tilde{\beta}}$, where $\tilde{\sigma}_{\tilde{\beta}}$ i a local sectionn of $\tilde{\mathcal{F}}$ that contains $\tilde{\beta(s)}$(see Theorem 6.2.9 [7]). Therefore $S \subset \sigma_{\beta(s)}$. On the other hand, it follows from Corollary 2.9 that $\exp_{\beta(s)}(\xi) \subset P_z$. Hence $\exp_{\beta(s)}(\xi) \subset P_z \cap S$. Now the statement follows from the fact that $P_z \cap S = \{z\}$.

This statement implies that the isometry $\varphi_{[\beta]}$ let the points of S fixed. Therefore $\varphi_{[\beta]}$ is a reflection in a totally geodesic hypersurface. Since $\varphi_{[\beta]}(x) \in L_x$, these reflections let $\mathcal{F} \cap \sigma$ invariant. $\square$

Proposition 3.4 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M. Suppose that the leaves are compact and that the holonomies of regular leaves are trivial. Let σ be a local section and Ω a connected component of the set obtained removing the singular stratification from the local section σ. Then :*

1) *an isometry $\varphi_{[\beta]}$ defined in Proposition 3.1 that let Ω invariant is the identity,*

2) *$\mathrm{Holsing}(\sigma)$ is generated by the reflections in the hypersurfaces of the singular stratification of the local section.*

Proof: a) Let p a point of Ω. Since the leaves are compact, L_p intercept Ω only a finite number of times. Hence, there exists a number n_0 such that $\varphi_{[\beta]}^{n_0}(p) = p$. Let $K := \{\varphi_{[\beta]}^i(p)\}_{0 \leq i < n_0} \subset \Omega$.

Lemma 3.5 *There exist only one ball $B_r(x) \supset K$ with minimal radio r The centre x of this ball belongs to K.*

Proof of the lemma. The proof of the lemma is standard, so we sketch the principal steps.

Statement 1: There exists a ball $B_r(x) \supset K$ with minimal radio. The centre x belongs to K.

This follows from the convexity of the balls.

Statement 2: A ball $B_r(x) \supset K$ with minimal radio is unique.

To check this statement suppose that there exists two balls $B_r(x_1)$ and $B_r(x_2)$ that contain K and have minimal radio r. Let x_3 be the middle point of the segment that joins x_1 to x_2. Then is possible to find a radio $\tilde{r} < r$, such that $B_{\tilde{r}}(x_3) \supset (B_r(x_1) \cap B_r(x_2))$ $\square$

Now we return to the proof of the item a) of the proposition.

Since $\varphi_{[\beta]}$ let K be invariant, $K = \varphi_{[\beta]}(K) \subset B_r(\varphi_{[\beta]}(x))$. Since $B_r(\varphi_{[\beta]}(x))$ is the ball with the minimal radio that contains K, then $\varphi_{[\beta]}$ fixes the point $x \in \Omega$. On the other hand, since the holonomy of regular leaves are trivial, $d_x\varphi_{[\beta]}$ is the identity. Since $\varphi_{[\beta]}$ is an isometry, it is the identity.

b) Let $\varphi_{[\beta]} \in Holsing(\sigma)$. We can compose $\varphi_{[\beta]}$ with reflections R_i's in the walls of the singular stratification such that $R_1 \circ \cdots \circ R_k \circ \varphi_{[\beta]}$ let Ω invariant and hence, it follows from the item a) that $R_1 \circ \cdots \circ R_k \circ \varphi_{[\beta]}$ is the identity. We conclude that $\mathrm{Holsing}(\sigma)$ is generated by the reflections in the hypersurfaces of the singular stratification. $\square$

Corollary 3.6 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M. Suppose that the leaves are compact and that the holonomies of regular leaves are trivial. Let $\mathrm{Tub}(L_q)$ be a tubular neighborhood of a leaf L_q, L_p a regular leaf that belongs to $\mathrm{Tub}(L_q)$ and $\Pi : \mathrm{Tub}(L_q) \to L_q$ the orthogonal projection. Then L_p is the total space of a fiber bundle with a projection Π, a basis L_q and a fiber that is diffeomorphic to an isoparametric submanifold of an euclidian space.*

Proof: $\Pi : L_p \to L_q$ is a submersion for the foliation is singular.

Statement $\Pi^{-1}(c) = \Sigma_c \cap L_p$ has only one connected component.

To check this statement suppose that $\tilde{L}_x, \tilde{L}_y \subset \Sigma \cap L_p$ are two disjoint leaves of $\Sigma \cap \mathcal{F}$. We can suppose that x and y belong to the same local section. Since $x, y \in L_p$, there exists $\varphi_{[\beta]} \in \mathrm{Holsing}(\sigma)$ such that $\varphi_{[\beta]}(x) = y$. The above corollary implies that $\varphi_{[\beta]}$ is a composition of reflections in the hypersurfaces of the singular stratification and hence $y = \varphi_{[\beta]}(x) \in \tilde{L}_x$. Therefore $\tilde{L}_x, \tilde{L}_y$ are the same leaf.

Now the proposition follows form the slice theorem and a theorem of Ehresmann [4], which we recall bellow.

Let $\Pi : L \to K$ a submersion, where L and K are compact manifolds. Suppose that $\Pi^{-1}(c)$ has only one connected component for each value c. Then the preimages are each other diffeomorphic and $\Pi : L \to K$ is the projection of a fiber bundle with total space L, basis K and fiber $\Pi^{-1}(c)$. $\square$

Proposition 3.7 *Let $\mathcal{F}$ be a s.r.f.s. on a complete riemannian manifold M, σ a local section and $p \in \sigma$. Then*

$$\overline{\mathrm{Holsing}(\sigma) \cdot p} = \overline{L_p \cap \sigma}.$$

In other words, the closure of $L_p \cap \sigma$ is an orbit of complete close pseudogroup of local isometries. In particular $\overline{L_p \cap \sigma}$ is a closed submanifold.

Proof: This result follows direct from results of E. Salem about pseudogroups of isometries (see appendix D in[6]).

One can argue like Salem (see Proposition 2.6 in [6]) to prove that $\overline{\mathrm{Holsing}(\sigma)}$ is complete and closed for the C^1 topology. It follows from Theorem 3.1 in [6] that a complete closed pseudogroup of isometry is a Lie pseudogroup. It also follows from E. Salem that a orbit of this Lie pseudogroup is a closed submanifold (see Corollary 3.3 in [6]). Therefore $\overline{\mathrm{Holsing}(\sigma)} \cdot p$ is a closed submanifold. Now it is easy to see that $\overline{\mathrm{Holsing}(\sigma)} \cdot p \supset \overline{\mathrm{Holsing}(\sigma) \cdot p}$. It is also easy to see that $\overline{\mathrm{Holsing}(\sigma)} \cdot p \subset \overline{\mathrm{Holsing}(\sigma) \cdot p}$. To finish the proof we have only to remember that $\mathrm{Holsing}(\sigma) \cdot p = L_p \cap \sigma$. $\square$

4 Examples

In this section we illustrate some properties of the singular holonomy constructing singular riemannian foliations with singularities by suspension of a homomorphism.

We start by recalling what a suspension is. For more details see for example the book of Molino [6][page 28,29; 96,97].

Let B and T be riemannian manifolds with dimension p and n respectively and let $\rho : \pi_1(B, b_0) \to Iso(T)$ be a homomorphism from the fundamental group of B to the group of isometries of T. Let $\hat{P} : \hat{B} \to B$ be the projection of the universal cover of B into B. Then we can define an action of $\pi_1(B, b_0)$ on $\widetilde{M} := \hat{B} \times T$ as

$$[\alpha] \cdot (\hat{b}, t) := ([\alpha] \cdot \hat{b}, \rho(\alpha^{-1}) \cdot t),$$

where $[\alpha] \cdot \hat{b}$ denote the deck transformation associated to $[\alpha]$ applied to a point $\hat{b} \in \hat{B}$.

We denote the set of orbits of this action by M and the canonical projection by $\Pi : \widetilde{M} \to M$. It's possible to see that M is a manifold. Indeed, given a simple open neighborhood $U_j \subset B$, we can construct the following bijection:

$$\begin{aligned} \Psi_j : \quad \Pi(\hat{P}^{-1}(U_j) \times T) &\to U_j \times T \\ \Pi(\hat{b}, t) &\to (\hat{P}(\hat{b}) \times t). \end{aligned}$$

If $U_i \cap U_j \neq \emptyset$ and connected, we can see that

$$\Psi_i \cap \Psi_j^{-1}(b, t) = (b, \rho([\alpha]^{-1})t)$$

for a fixed $[\alpha]$.

So there exists an unique manifold structure on M for which Ψ_j are local diffeomorphisms. We define a map P as

$$\begin{aligned} P : \quad M &\to B \\ \Pi(\hat{b}, t) &\to \hat{P}(\hat{b}) \end{aligned}$$

It's possible to see that M is a total space of a fiber bundle, P is the projection of this fiber bundle, T is the fiber, B is the basis and the image of ρ is the structure group.

At last we define $\mathcal{F} := \{\Pi(\hat{B}, t)\}$, i.e., the projection of the trivial foliation defined as the product of $\hat{B}$ with each t. It is possible to see that this is a foliation transverse to the fibers of the fiber bundle. In addition this foliation is a riemannian foliation such that the transversal metric coincide with the metric of T.

Example 4.1 In what follows we construct a singular riemannian foliation with sections such that the intersection of a local section with the closure of a regular leaf is an orbit of an action of a subgroup of isometries of the local section. This illustrates Proposition 3.7.

Let T denote the product $\mathbf{R}^2 \times \mathbf{S}^1$ and $\hat{\mathcal{F}}_0$ the singular foliation of codimension 2 on T such that each leaf is the product of a point of $\mathbf{S}^1$ with a circle in $\mathbf{R}^2$ whose centre is $(0,0)$. It is easy to see that the foliation $\hat{\mathcal{F}}_0$ is a singular riemannian foliation with sections and that sections are cylinders. Let B be the circle $\mathbf{S}^1$ and q be a irrational number. Then we define the homomorphism ρ as

$$\begin{aligned} \rho: \quad \pi_1(B, b_0) &\rightarrow \text{Iso}(T) \\ n &\rightarrow ((x, s) \rightarrow (x, \exp(i\, n\, q) \cdot s)). \end{aligned}$$

Finally we define $\mathcal{F} := \Pi(\hat{B} \times \hat{\mathcal{F}}_0)$. One can notice that $\mathcal{F}$ is a singular riemannian foliation with sections such that the intersection of each section with the closure of a regular leaf is an orbit of an isometric action on the section. Indeed one can see this action as translations along the meridians of a cylinder, which is a section of the foliation.

Example 4.2 In what follows we construct a singular riemannian foliation with sections such that $\text{Holsing}(\sigma)$ has an element that can not be generated by the reflections in the hypersurfaces of the singular stratification.

Let T be a compact Lie group (e.g. $T = SU(3)$) and a manifold B such that $\pi_1(B) = \mathbf{Z}_2$ (e.g. $B = SO(n)$). We define the homomorphism ρ as follows

$$\begin{aligned} \rho: \quad \pi_1(B, b_0) &\rightarrow \text{Iso}(T) \\ 0 &\rightarrow (t \rightarrow t) \\ 1 &\rightarrow (t \rightarrow t^{-1}). \end{aligned}$$

Let us consider the action of T on itself by conjugation, i.e. $t \cdot g := t \, g \, t^{-1}$. The orbits of this action are leaves of a singular riemannian foliation that has tori as sections. We denote this singular foliation by $\hat{F}_0$. It's easy to see that $(T \cdot g)^{-1} = T \cdot g^{-1}$. This assure us that $\mathcal{F} := \Pi(\hat{B} \times \hat{\mathcal{F}}_0)$ is a singular foliation on M. We can give a metric to M such that the metric of the fibers coincide with the metric of T. Then $\mathcal{F}$ turns out to be a singular riemannian foliation whose sections are contained in the fibers. This sections are tori.

Now it's possible to see that the leaves of $\mathcal{F}$ intersect a Wely chamber of each torus in more then one point. In fact give a point x_1 belonging to a Wely chamber, we can reflect it in the walls of the singular stratification and get another point x_2 belonging to another Wely chamber and such that x_2^{-1} belongs to the same Wely chamber of x_1. Since inverse points belong to the same leaf, x_2^{-1} belongs to the same leaf of x_1.

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